

The Quantum Locker Puzzle

David Avis
McGill University
avis@cs.mcgill.ca

Anne Broadbent
IQC, University of Waterloo
albroadb@iqc.ca

Abstract

The locker puzzle is a game played by multiple players against a referee. It has been previously shown that the best strategy that exists cannot succeed with probability greater than $1 - \ln 2 \approx 0.31$, no matter how many players are involved. Our contribution is to show that quantum players can do much better—they can succeed with probability 1. By making the rules of the game significantly stricter, we show a scenario where the quantum players still succeed perfectly, while the classical players win with vanishing probability. Other variants of the locker puzzle are considered, as well as a cheating referee.

Keywords: *quantum complexity, Grover search, locker puzzle*

1. Introduction

Grover’s quantum algorithm [7] provides a quadratic speedup over the best possible classical algorithm for the problem of unsorted searching in the query model. While Grover’s search method has been shown to be optimal [1], our results reveal that in the context of multi-player query games, applying Grover’s algorithm yields success probabilities that are much better than the success probabilities of classical optimal protocols. Specifically, we show that in the case of the *locker puzzle*, quantum players succeed with probability 1 while the known optimal classical success probability is bounded above by $1 - \ln 2 \approx 0.31$. In order to amplify this separation, we prove that a significantly stricter version of the locker puzzle has vanishing classical success probability, while still admitting a perfect quantum strategy. We also consider the empty locker and the coloured slips versions of the locker puzzle, and the possibility of a cheating referee.

2. The Locker Puzzle

The *locker puzzle*¹ is a cooperative game between a team of n players numbered $1, 2, \dots, n$ and a referee. In the initial phase of the game, the referee chooses a random permutation σ of $1, 2, \dots, n$, and for each player i she places number i in locker $\sigma(i)$. In the following phase, each player is individually admitted into the locker room. Once in the room, each player is allowed to open $n/2$ lockers, one at a time, and look at their contents (for simplicity, we’ll take n to be even). After the player leaves the room, all lockers are closed. The players are initially allowed to discuss strategy, but once the game starts, they are separated and cannot communicate. An individual player i *wins* if he opens a locker containing number i , while the team of n players *wins* if all individual players win.

We would like to know what is the best strategy for the team of n players. A naïve approach is for each player to independently choose $n/2$ lockers to open. Each player wins independently with probability $1/2$, hence the team wins with probability $1/2^n$. Surprisingly, it is known that the players can do much, much better! We will review in Section 2.1 an optimum strategy by which, for any n , the players can win with probability at least 0.30685.

The locker puzzle was originally considered by Peter Bro Miltersen, and was first published in [4]; a journal version appears in [5]. Sven Skylum is credited for the pointer-following strategy that we will give in the next section. A proof of optimality for this strategy is given by Eugene Curtin and Max Warshauer [3]. Our presentation of the classical puzzle and its solution follows along the lines of their article. Many variations have been proposed [6]. We will consider the variations of *empty lockers* in Section 4.1, *coloured slips* in Section 4.2 (to be accurate, the locker and the coloured slips puzzles are variants of the empty locker puzzle), and a *cheating referee* in Section 4.3.

¹The *locker puzzle* also sometimes refers to another scenario that involves the opening and closing of locker doors in a hallway, where the question is: after a specific series of moves, which locker doors remain open? Our puzzle here is different (and much more challenging).

2.1. An Optimal Classical Solution

We saw that a naïve solution allows the players to win with an exponentially small probability. How can we devise a strategy that does better? The reader avid to search for a solution on his or her own is encouraged to do so now.

The key is to find a solution where the individual success probabilities are not independent. Consider the following strategy: when first entering the locker room, player i opens locker number i . A number is revealed; this is used to indicate which locker to open next (i.e. if number j is revealed, the next locker opened is locker j). Each player executes this pointer-following strategy until $n/2$ lockers are opened.

To analyze the success probability, note that the team will win provided that the *permutation* that corresponds to the placement of numbers in lockers by the referee does not contain a cycle of length longer than $n/2$. The probability of such a long cycle occurring is:

$$\sum_{k=1}^{n/2} \frac{1}{n/2 + k}. \quad (1)$$

It can be shown that as $n \rightarrow \infty$, $\sum_{k=1}^{n/2} \frac{1}{n/2 + k} \rightarrow \ln 2$ and that the sum increases with n . Hence the probability that the team wins is decreasing to $1 - \ln 2 \approx 0.30685$.

Using a reduction to another game, this strategy can be shown to be optimal [3].

3. A Quantum Solution

We now present our first contribution: a quantum solution to the locker puzzle, which performs better than the classical solution.

As before the referee chooses a random permutation σ and she places numbers in the lockers according to this permutation. In the quantum solution, we allow the players to open locker doors in *superposition*, each player working with his own quantum register. This is analogous to the quantum query model. For the quantum case, we need to modify the goal of the game which, for player i , becomes to *correctly guess* the locker containing number i after $n/2$ queries, and *not* to open locker containing number i , because this would be too easy to do in superposition! We show that quantum players can always win at the locker game. In fact, our results are stronger: we give a stricter version of the locker puzzle for which the optimal classical solution succeeds with vanishing probability, while a quantum strategy always succeeds!

3.1. Improving the Success Probability

The main idea is to apply Grover's quantum search algorithm to the locker puzzle. For player i , we consider the action of opening a locker as a query to the oracle which when input locker number x , $1 \leq x \leq n$, outputs the following:

$$f_i(x) = \begin{cases} 1 & \text{if } \sigma(i) = x \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

Note that this oracle is weaker than the oracle in the original puzzle which would output $f_i(x) = \sigma^{-1}(x)$. We discuss this further in Section 3.3 and in the conclusion.

Grover's search algorithm [7] was thoroughly analyzed in [1], where it was shown that in a black-box search scenario where it is known that a single solution exists, $\frac{\pi}{4}\sqrt{n}$ queries yield a failure probability no greater than $\frac{1}{n}$, where n is the number of elements in the search space (here, n is assumed to be large). This was further improved in [2], where it was shown that the same amount of queries is sufficient to find a solution with *certainty*.

Applying this directly to the quantum players of the locker puzzle yields the following:

1. Each player performs $\frac{\pi}{4}\sqrt{n}$ queries (this is less than the $\frac{n}{2}$ queries in the classical solution).
2. Each player wins independently with certainty, implying that the team wins with certainty.

3.2. Reducing the Number of Queries

We've seen that quantum players of the locker game can succeed with probability 1. Our solution only requires $\frac{\pi}{4}\sqrt{n}$ oracle queries per player. Hence, we now consider the asymptotically stricter version of the locker puzzle, where players are allowed to open at most \sqrt{n} lockers. The next theorem states that the success probability for classical players goes quickly to 0.

Theorem 3.1. *In the locker puzzle with \sqrt{n} queries, classical players win with probability at most $\frac{1}{\lfloor \sqrt{n} \rfloor!}$.*

Proof. Let $N = \lfloor \sqrt{n} \rfloor$. We upper bound the success probability of the first N^2 players, when each player is allowed to open N lockers. Since $n \geq N^2$, this upper bounds the success probability of all n players.

Consider a new game where the first player opens exactly N lockers and publicly reveals all of their contents. If the first player's number is not revealed the players lose and the game is over. Otherwise the N revealed players have successfully located their lockers. These N lockers and players are now removed from the game. The first player has success probability at most N/N^2 .

In successive rounds, a player is chosen from amongst those not yet removed from the game. He continues in the same way by choosing N of the remaining lockers and revealing their contents. If he finds his label, again N lockers and players are removed from the game. The game stops whenever a chosen player does not find his label. Otherwise it continues for N rounds and terminates with a win for the players.

The success probability of the new game is at most

$$\frac{N}{N^2} \cdot \frac{N}{N^2 - N} \cdot \frac{N}{N^2 - 2N} \cdot \dots \cdot \frac{N}{N} = \frac{1}{N!}. \quad (3)$$

The original game with no revealing of numbers cannot do better. \square

3.3. Optimality and Oracle Strength

Theorem 3.2. *In the quantum query model with oracle (2) the total number of queries required to obtain a success probability of one for the players is in $\Omega(n\sqrt{n})$.*

Proof. First consider a variation of the quantum game where the players act sequentially in the order $1, 2, \dots, n$ and are allowed to announce their results to the other players. The number of queries performed by Player 1 must be in $\Omega(\sqrt{n})$ or he will not succeed with probability one. This follows from the analysis of Grover's algorithm, see [1].

The only information given by the oracle f_1 is the location of the locker containing label 1. Suppose player 2 is allowed to receive this information and remove that locker from consideration. The permutation σ induces a random permutation on the remaining $n - 1$ lockers. Player 2's success probability is then one only if his number of queries is in $\Omega(\sqrt{n - 1})$. Continuing, the i -th player must ask a number of queries in $\Omega(\sqrt{n - i})$. The total number of queries is therefore in $\Omega(n\sqrt{n})$.

In the modified game we share all information available to all players that have not already played. So this shows a lower bound of the same order for the original version of the quantum game where no information is shared. \square

Let us now compare the strength of oracle (2) with the stronger oracle where $f_i(x) = \sigma^{-1}(x)$. In the classical setup, the weaker oracle (2) merely tells a given player whether or not his label is in a requested locker. There are an even number n of lockers and he can ask $t = n/2$ queries. Again we consider a sequential version of the game as described above, where each player reveals his results. If he succeeds, he reveals the locker with his number and that locker is removed. For the other lockers he queried, the only information he has is that they did not contain his label. Therefore after his locker is removed, the other players have no further information. The success probability of this

variation of the locker game is:

$$\frac{t}{2t} \cdot \frac{t}{2t-1} \cdot \frac{t}{2t-2} \cdot \dots \cdot \frac{t}{t+1} = \frac{t^t t!}{(2t)!} \quad (4)$$

$$\approx \frac{t^t \sqrt{2\pi t} t^t e^{-t}}{\sqrt{2\pi 2t} (2t)^{2t} e^{-2t}} = \frac{1}{\sqrt{2}} \left(\frac{e}{4}\right)^t \approx \frac{1}{\sqrt{2}} 0.824^n, \quad (5)$$

where we have used Stirling's formula twice. This is exponentially small and provides an upper bound on the success probability of the classical locker game with the weak oracle (2). By comparison, as we saw in Section 2.1 the players can win with constant probability using the stronger oracle. An open question is whether the quantum algorithm can be improved by using this stronger oracle.

4. Variants of the Locker Puzzle

The original motivation for the locker puzzle came from the study of time-space tradeoffs for the substring search problem in the context of *bit probe complexity* [4]. There, a version with both *empty lockers* and *coloured slips* was presented. We now examine these two variations separately and consider the quantum case.

4.1. Empty Lockers

Suppose there are a total of $b \geq n$ lockers. The referee selects an unordered subset σ of $\{1, \dots, b\}$ with cardinality n and she puts label i into locker $\sigma(i)$ for $i = 1, \dots, n$. The remaining $b - n$ lockers are empty. Assume b is even, and we allow the players to open up to $b/2$ lockers. An optimum winning strategy for this more general situation is unknown: the pointer algorithm fails if an empty locker is opened. Even for the case $b = 2n$, where half of the lockers are empty, it is still unknown if there is a classical strategy with success probability bounded away from zero [6]. However, the quantum strategy given in Section 3 still succeeds with probability one with a number of queries in $O(\sqrt{b})$ per player, for a total of $O(n\sqrt{b})$ queries. It suffices to modify the oracle (2) so that x runs over the range $1 \leq x \leq b$, and query it $\frac{\pi}{4}\sqrt{b}$ times. If it turns out that for these same parameters, the classical success probability vanishes, then the power of the quantum world would be once more confirmed, as in Section 3.2. and Section 3.3.

4.2. Coloured Slips

Consider the empty lockers game with $b \geq n$ lockers, again with n players and n slips of paper, each labelled $1, \dots, n$. This time the referee colours each slip either red or blue as she chooses, and places them in a randomly selected subset of n lockers. As before, each player i may open $b/2$ lockers using any adaptive strategy, and based

on this, must make a guess about the colour of the slip labelled i . The players win if every player correctly announces the colour of his slip. With $b = n$, this can be solved with the pointer-following algorithm and the players have success probability about 0.31.

In the quantum setting, the players can win with probability one at the colour guessing game also, by changing the oracle (2). Let $c(i)$ be the colour of the slip for player i . Define for $1 \leq x \leq b$ and $1 \leq i \leq n$:

$$g_i(x) = \begin{cases} 1 & \text{if } \sigma(x) = i \text{ and } c(i) = \text{red} \\ 0 & \text{otherwise.} \end{cases} \quad (6)$$

Now we use the protocol described in Section 3.1 with each player querying this new oracle $\frac{\pi}{4}\sqrt{b}$ times. If for player i $c(i) = \text{red}$, then there is exactly one x for which $g_i(x) = 1$ and Grover's algorithm returns $x = \sigma(i)$ with probability one. Otherwise, if $c(i) = \text{blue}$ then g_i is identically zero and Grover's algorithm may return any value x . The player now makes one further call to oracle (6) with the returned value x and guesses red if the oracle returns one and blue otherwise.

4.3. Cheating Referee

A cheating referee can obviously beat the players in the locker game. She simply has to omit the label of one of the players. This could be easily exposed by requiring that all the lockers be opened and checked at the end of the game.

A more subtle way of cheating is if the referee can somehow choose the permutation σ . In the original locker game, let $s = n/2 + 2$, and let i_1, \dots, i_s be a random unordered subset of s players. She may set $\sigma(i_1) = i_s$, $\sigma(i_{j+1}) = i_j$, $j = 1, \dots, s-1$, and fill out the rest of σ at random from the remaining players. It is easy to verify that, using the pointer algorithm, player i_1 opens $n/2$ lockers i_1, \dots, i_{s-2} and does not find his label. He has to guess and loses with probability about $2/n$. The same thing happens for each of the players i_j . (Incidentally, the reason for not choosing $s = n/2 + 1$ is that the players not finding their label may guess the locker number they see in the last locker they open, winning the game with probability one!). Using variants of this idea the referee may cheat successfully for some time before the players catch on. If the players have access to shared randomness (which is unknown to the referee), they can circumvent this problem by first applying their own permutation on the lockers before opening any of them. Interestingly, our quantum protocol is impervious to a referee who maliciously chooses the permutation, and does not require shared randomness.

5. Conclusion and Discussion

It was previously known that the locker puzzle has an intriguing classical optimal solution. Now we know that the locker puzzle and its variants also have interesting quantum solutions which perform significantly better than the classical ones.

We have given a quantum solution in the black-box query complexity model that *does not use the pointer-following technique that is crucial to the classical optimal solution*. It would be interesting to see if using the stronger classical oracle could lead to a quantum solution that works with a reasonable probability of success using $o(n\sqrt{n})$ total queries. With this stronger oracle, perhaps shared entanglement could help the players? It would also be interesting to see if, analogous to the classical case, our results have any consequences for time-space tradeoffs for data structures [6].

Acknowledgements

We would like to thank Bruce Reed for introducing us to the classical version of the locker puzzle and Richard Cleve for pointing out the perfect quantum search of [2]. This work was partially supported by an NSERC discovery grant and an NSERC postdoctoral fellowship.

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